ON SIMILARITY LAWS FOR THE DEVELOPED TURBULENCE OF DILUTE POLYMER SOLUTIONS

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The peculiarities of turbulence in weak polymer solutions are discussed from the aspects of similarity theory and dimensional analysis. Only the papers [1-67] are mentioned in which the flow of polymer solutions in tubes was discussed, because of the lack of space. Many of the questions under discussion were touched upon earlier in [26], as well as in [3, 32, 47].

Velocity Defect Law. A flow domain exists far from the walls (exterior) in a developed turbulent stream, whose integral characteristics are independent of the fluid properties (the fluid can be considered ideal) and the wall roughness but are determined by the natural length and velocity scales.†

These scales in a steady flow in a tube are determined, in turn, by the tube radius r, the distance to the wall z, and the momentum flux to it u_*^2 so that we have for the deviation‡ of its mean velocity from the maximum value U

$$U - \langle u(z) \rangle = u_* f_1(\eta), \quad \eta \gg l/r,$$
$$f_1(1) = f_1'(1) = 0. \tag{1}$$

Here $f_1(\eta)$ is the universal dimensionless function $\eta \equiv z/r$, l is the set of lengths of "molecular" influence.

The relationship (1), known as the "velocity defect law," or the "external law" of turbulent flow similarity, is confirmed well for both a viscous fluid as well as for weak polymer solutions [8, 9, 16, 18, 24, 27, 38, 53, 57].

An intermediate domain ($r \gg z \gg l$) exists in developed turbulence, in which an isolated length scale is absent and a change in scale can result only in a parallel shift of the velocity profile. The unique continuous distribution of this kind is the logarithmic, so that we can write

$$f_1(\eta) = -A_1 \ln \eta + B_1 - w(\eta), \quad \eta \gg l/r, \tag{2}$$

$w(\eta) = 0$ for $\eta \leq \eta_0$.

According to measurements in a viscous fluid $\eta_0 \approx 0.25$, $A_1 \approx 2.5$, $B_1 \approx 0.7$. For weak solutions in "large" tubes, (2) also holds [8, 9, 15-17, 24, 25, 31, 32, 36, 40, 41, 42, 44, 51, 57-60, 67, 38].

The Wall Law. The velocity distribution in the "internal" near-wall region ($z \ll r$), on the other hand, is independent of the external scales similar to r but can depend on the "molecular" properties of the fluid and on the wall structure.

The governing parameters for the flow near a smooth wall are z, u_* and the viscosity ν in a viscous fluid. Their other characteristics must also be taken into account in the description of solutions.

†The fluid is considered incompressible and the system of measurement units is such that the density is $\rho = 1$.

‡The acceleration, rather than the absolute value of the velocity, is meaningful in an ideal fluid.

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Experiments show that many characteristics of solutions are essential to reduce the drag: the concentration and molecular-weight distribution of the polymer molecules, the type of solvent, the branchiness and stiffness of the polymer chains, the pH of the solution, etc.

Let us assume that all these quantities can be taken into account by using the additional length parameter l (or the time† θ) and a set of dimensionless parameters p (concentration, say) independent of the magnitude of the viscosity. Then we obtain for the velocity profile at a smooth wall

$$\langle u^+ \rangle = f(z^+, l^+, p), \quad z^+ \leqslant r^+, \qquad f(0, \ldots) = 0, \tag{3}$$

which compares the content of the "internal law" of similarity to the "wall law." Here f is the universal dimensionless function and the cross marks the variables nondimensionalized by using‡ u_* and ν. In the intermediate domain, we have as before

$$f(z^+, l^+, p) = A_1 \ln z^+ + B(l^+, p), \quad r^+ \gg z^+ \gg 1, \; l^+. \tag{4}$$

Measurements yield $B = B_0 \approx 5.5$.

Maximal Velocity. When there is a "logarithmic layer" (2), (4) in developed turbulence, it follows from (1)-(4)

$$U^+ = A_1 \ln r^+ + B_1 + B(l^+, p). \tag{5}$$

Introducing the Reynolds number $Re^0 = U^+r^+$ and the drag coefficient $c_f = 2(U^+)^{-2}$, this relationship can be rewritten as

$$\sqrt{2/c_f} = A_1 \ln (Re^0 \sqrt{c_f/2}) + B_1 + B(lr^{-1} Re^0 \sqrt{c_f/2}, p). \tag{6}$$

If the length $\bar{l}$, normalized in such a way that $\bar{l}^+ = 1$ would correspond to the beginning of the influence of the polymer on the turbulence is introduced in place of l, then we obtain for the number R^0_{cr} for which the drag reduction starts

$$Re^0_{cr} = \frac{r}{L}\left(A_1 \ln \frac{r}{L} + B_0 + B_1\right), \quad L = \begin{cases} \bar{l}, & (l), \\ \sqrt{\nu\bar{\theta}}, & (\theta) \end{cases} \tag{7}$$

for the length and time parameters, respectively.

Approximate Description of the Velocity Profile. If the details of distributions close to the wall are neglected, then the influence of the polymer reduces to high filling of the profile because of "slip at the wall." Such a single layer description has been discussed in [2, 5, 16, 30, 52]. Its accuracy is certainly not high.

It is assumed that the velocity distribution near the wall has the same linear form for weak solutions as for a viscous fluid. The majority of the measurements [24, 25, 38, 41, 51, 53, 55, 56, 57, 63] favor such an assumption, and hence, a two-layer description (see [8, 12, 59] for example)

$$f(z^+, l^+, p) = \begin{cases} z^+, & z^+ \leqslant \Delta^+, \\ A_1 \ln z^+ + B(l^+, p), & \Delta^+ \leqslant z^+ \leqslant \eta_0 r^+, \end{cases} \tag{8}$$

$$\Delta^+ = A_1 \ln \Delta^+ + B(l^+, p) \tag{9}$$

is often used for the interior domain. In a viscous fluid $\Delta^+ = \Delta_0^+ \approx 11.6$ and the accuracy of the description is not satisfactory only in the transition zone $\Delta_0^+/3 \leqslant z^+ \leqslant 3\Delta_0^+$ (Fig. 1). Similarly in polymer solutions with slight drag reduction.

For a large drag reduction ($\bar{l}^+ \gg 1$) the Δ^+ increases significantly and the transition zone in whose subregion $1 \ll z^+ \ll l^+$ there is no preferred length scale and therefore the velocity distribution is almost logarithmic, is expanded strongly. The same should be expected for the dependence of B on l^+. The location of the viscous sublayer boundary Δ_1 for $l^+ \gg 1$ can depend on ν and on p but not on l, and conversely, the location of the outer boundary Δ_2 should be independent of the viscosity.

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†By using u_* the quantities with dimensionality of the length θu_* can be compared to quantities with the dimensionality of the time θ in a turbulent shear stream, and conversely.

‡The value of the viscosity on the wall is used. Such an approximation is satisfactory if the change in viscosity within the viscous sublayer and the transition zone is small. This is visibly so in weak solutions.

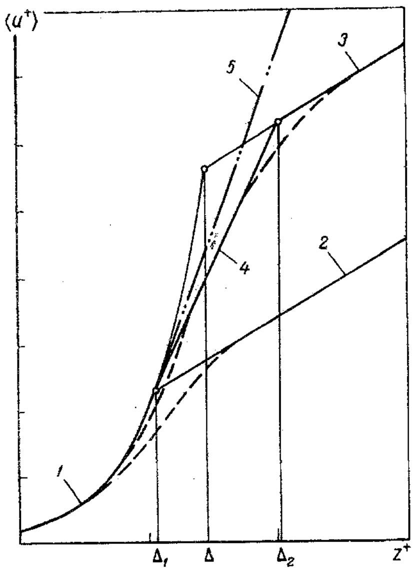

Fig. 1

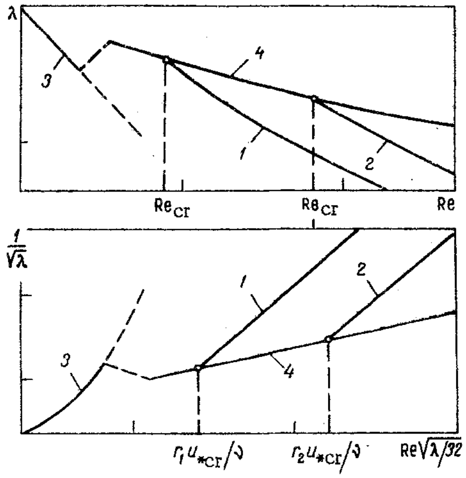

Fig. 2

Fig. 1. Model representation of the mean velocity profile: 1) $\langle u^+\rangle = z^+$; 2) $\langle u^+\rangle = A_1 \ln z^+ + B_0$; 3) $\langle u^+\rangle = A_1 \ln z^+ + B_0 + \Delta B$; 4) $\langle u^+\rangle = A_2 \ln z^+ + B_2$; 5) $\langle u^+\rangle = 11.6 \ln z^+ - 16.8$. The dashed lines show the behavior of the true profile near the characteristic points Δ_1 and Δ_2.

Fig. 2. Drag curves: 1, 2) for solutions in tubes with radii r_1 and r_2; 3) laminar flow; 4) turbulent viscous fluid flow.

Using all these asymptotic dependences for $l^+ \sim 1$, we obtain the following three-layer description of the velocity profile at the wall (Fig. 1)

$$f(z^+, l^+, p) = \begin{cases} z^+, & z^+ \leqslant \Delta_1^+, \\ A_2(p) \ln z^+ + B_2(p), & \\ A_1 \ln z^+ + B(l^+, p), & \Delta_2^+ \leqslant z^+, \end{cases} \tag{10}$$

$$B - B_0 \equiv \Delta B = \begin{cases} 0, & \bar{l}^+ < 1, \\ A_3(p) \ln \bar{l}^+, & \bar{l}^+ \geqslant 1, \end{cases} \tag{11}$$

where we have by requiring that this model reduce for $\bar{l}^+ \to 1$ to the two-layer model for a viscous fluid (8):

$$\Delta_1^+ = \Delta_0^+ \approx 11.6; \quad \Delta_2^+ \approx 11.6\bar{l}^+, \quad \bar{l}^+ \geqslant 1. \tag{12}$$

The conditions for continuity of the distribution (10) impose the constraints

$$A_2 = A_1 + A_3, \quad B_2 = 11.6 - 2.45 A_2. \tag{13}$$

Therefore, the form of the velocity profile for a solution in a three-layer approximation is determined completely by the two additional parameters $A_3(p)$ and $\bar{l}^+$.

Found empirically in [12] was the formula

$$\Delta B = \begin{cases} 0, & u_* < u_{*\mathrm{cr}}, \\ \alpha \lg (u_*/u_{*\mathrm{cr}}), & u_* > u_{*\mathrm{cr}}, \end{cases} \tag{14}$$

which is obtained from (11) if the following notation is introduced

$$\alpha = 2.3 A_3, \quad u_{*\mathrm{cr}} = \nu \bar{l}^{-1}, \quad (l), \tag{15}$$

$$\alpha = 4.6 A_3, \quad u_{*\mathrm{cr}} = \sqrt{\nu \bar{\theta}^{-1}}, \quad (\theta) \tag{16}$$

for the length and time parameters, respectively. A formula of the type (14) was used for the time parameter in [8].

The general considerations do not exclude the possibility of saturation of the dependence of B on l^+ for $l^+ \gg 1$. Found empirically in [61] is the formula

$$\Delta B = \Delta B_{\max} \frac{u_*^2/\tau_{WC}}{1 + u_*^2/\tau_{WC}}, \tag{17}$$

for which the intermediate asymptotic is the logarithmic dependence presented above, where

$$\alpha \approx 1.15 \Delta B_{\max}, \quad u_{*\mathrm{cr}} \approx 0.37 \sqrt{\tau_{WC}}.$$

The three-layer approximation (10) is often used [42, 43, 46, 52], hence it was considered in [46] that A_2 can take on any value depending on the solution, and the coefficient A_2, similarly to A_1, was declared a universal constant in [42, 43] and $A_2 \approx 10$ was found in [42] and $A_2 \approx 11.7$ in [43].

If the absence of inflection point† on the mean velocity profile (10) is required, then

$$A_1 < A_2 < 11.6. \tag{18}$$

For $A_2 = 11.6$ the three-layer description (10) is analogous to the known Karman model. A somewhat different generalization of the Karman model was used in [28, 60].

Drag Reduction. Integrating the velocity distribution with respect to the tube cross-section for developed turbulence when $\Delta_2 \ll r$, we find

$$U^+ - \langle \bar{u}^+ \rangle = C \approx B_1 + \frac{3}{2} A_1 - 2 \int_{\eta_*}^{1} w(\eta)(1-\eta)\, d\eta, \quad \Delta_2 \ll r; \tag{19}$$

here C is independent of the fluid properties and $C \approx 4.1$ according to measurements in a viscous fluid.

Eliminating U^+ from (5), (19) and introducing the drag coefficient $\lambda = 8u_*^2/\langle \bar{u} \rangle^2$ and the Reynolds number $\mathrm{Re} = 2r\langle \bar{u} \rangle \nu^{-1}$, we obtain

$$\sqrt{8/\lambda} = A_1 \ln(\mathrm{Re}\sqrt{\lambda/32}) + 2.1 + \Delta B, \tag{20}$$

where $\Delta B = 0$ for $\mathrm{Re} < \mathrm{Re}_{\mathrm{cr}}$ in conformity with (13) and

$$\Delta B = A_3 \ln(\mathrm{Re}\sqrt{\lambda/32}) - A_3 \ln(r/\bar{l}),$$
$$\Delta B = 2A_3 \ln(\mathrm{Re}\sqrt{\lambda/32}) - A_3 \ln(r^2/\nu\bar{\theta}), \quad \mathrm{Re} \gg \mathrm{Re}_{\mathrm{cr}} \tag{21}$$

for the length and time parameters, respectively. In both cases, it is also possible to write (see (14)-(16))

$$\Delta B = \alpha \lg(\mathrm{Re}\sqrt{\lambda/32}) - \alpha \lg(r u_{*\mathrm{cr}}/\nu), \quad \mathrm{Re} \gg \mathrm{Re}_{\mathrm{cr}}. \tag{22}$$

It is seen from (20)-(22) how the drag depends on the tube radius, and in particular (Fig. 2)

$$\mathrm{Re}_{\mathrm{cr}} = \frac{2r}{L}\left(A_1 \ln \frac{r}{L} + 2.1\right), \quad L = \frac{\nu}{u_{*\mathrm{cr}}} = \begin{cases} \bar{l} \\ \sqrt{\nu\bar{\theta}} \end{cases}. \tag{23}$$

The identical stress on the wall $u_{*\mathrm{cr}}^2$ corresponds to the beginning of drag reduction in tubes of diverse diameters for the same solution.

Such singularities in the drag reduction are characteristic for all polymer solutions. The effect of the diameter had already been noted in [1, 3, 4, 7]. The critical stress of the beginning of the effect had been detected [8, 10, 12, 13, 14]. The two-parameter dependence of the effect on α and $u_{*\mathrm{cr}}$ is verified well.

Formula (9) can be given the form

$$U/\langle \bar{u} \rangle = 1 + C\sqrt{\lambda/8}.$$

The deductions about the change in $U/\langle \bar{u} \rangle$ from experimental investigations [31-33, 37, 62] are contradictory.

A deviation from dependences of the type (20)-(22) occurs principally at high Reynolds numbers. It has been established that in the majority of cases this is related to the degradation of solutions of macromolecular substances in a turbulent stream. This is beyond the scope of the analysis presented above.

†In the presence of inflection points on the mean velocity profile, the question of the stability of such a flow arises.

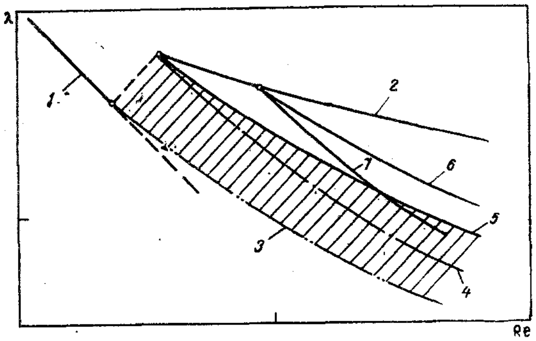

Fig. 3

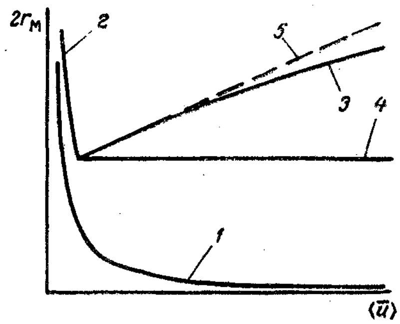

Fig. 4

Fig. 3. Drag curves. 1) $\lambda = 64/\mathrm{Re}$; 2) $\sqrt{8/\lambda} = 2.5 \ln(\mathrm{Re}\sqrt{\lambda/32}) + 2.1$; 3) limit asymptote; 4) from (27); 5) from (26); 6) $\sqrt{8/\lambda} = A_2 \ln(\mathrm{Re}\sqrt{\lambda/32}) + 2.1 - A_3 \ln(r/l)$; 7) $\sqrt{8/\lambda} = (A_1 + 2A_3)\ln(\mathrm{Re}\sqrt{\lambda/32}) + 2.1 - 2A_3 \ln(r/\sqrt{\nu\bar{\theta}})$. The domain for "small" tubes is cross-hatched.

Fig. 4. Dependence of the minimum diameter $2r_M$ on $\langle\bar{u}\rangle$ for 1) viscous solvent; 2) solution flowing without drag reduction; 3, 4) solution whose drag reduction is characterized by the parameter θ or l, respectively; 5) $2r_M = 17.6\bar{\theta}\langle\bar{u}\rangle$.

Minimum Diameter. Up to now we have spoken about developed turbulence, i.e., flows in tubes of such large diameters and for such high Reynolds numbers that the external sublayer of the turbulent core and the near-wall zone do not overlap

$$\eta_0 r \gg \Delta_2$$

and a logarithmic zone with a distribution of the type (2), (4) exists.

It is not taken into account in the three-layer description (10) that a smooth change in the true velocity profile occurs near $z^+ = \Delta_2^+$ for two-three Δ_2^+. Taking this into account, let us select $2.5\Delta_2$ for the boundary of the transition zone and then by using (12) we obtain the following estimate for the minimum diameter for which speaking about the usual logarithmic zone still has sense:

$$2r_M^+ \approx \begin{cases} 250, & \bar{l}^+ < 1, \\ 250\bar{l}^+, & \bar{l}^+ > 1, \end{cases} \tag{24}$$

which can be rewritten for the Reynolds number

$$\mathrm{Re}_M \approx \begin{cases} 3.5\cdot 10^3, & \bar{l}^+ < 1, \\ (3.5\cdot 10^3 + 250 A_2 \ln \bar{l}^+)\,\bar{l}^+, & \bar{l}^+ > 1. \end{cases} \tag{25}$$

Hence, it is seen that in a viscous fluid, as well as for a flow without drag reduction ($\mathrm{Re} < \mathrm{Re}_{cr}$), this estimate agrees with the upper value of the Reynolds number for the transition from the laminar to the turbulent flow modes.

On the (λ, Re) plane the curve

$$\sqrt{8/\lambda_M} = A_2(p)\ln(\mathrm{Re}_M \sqrt{\lambda_M/32}) + 2.1 - 4.8A_3(p) \tag{26}$$

separates the domain of "small" tubes (hatched in Fig. 3) from the domain of "large" tubes for a given polymer solution. If (18) is taken, then the steepest curve of this kind will be

$$\sqrt{8/\lambda} = 11.6 \ln(\mathrm{Re}\sqrt{\lambda/32}) - 42 \tag{27}$$

(the dash-dot curve in Fig. 3).

Formula (25) shows that the "minimum diameter" in a viscous fluid and in solutions with $\mathrm{Re} < \mathrm{Re}_{cr}$ diminishes as the velocity increases

$$2r_M \approx 3500\nu/\langle\bar{u}\rangle. \tag{28}$$

If the length l is responsible for the drag reduction, then it follows from (24)

$$2r_M \approx 250\bar{l} = 250\nu/u_{*cr}. \tag{29}$$

If the time parameter plays the principal part, then

$$2r_M \approx 250\bar{\theta}u_* = 250\nu u_*/u_{*cr}^2, \quad u_* \geqslant u_{*cr} \tag{30}$$

or expressing u_* in terms of the mean velocity

$$2r_M \approx 250x\sqrt{\nu\theta},\quad x(14.2 + 2A_3 \ln x) = \langle \bar{u} \rangle \sqrt{\theta/\nu}. \tag{31}$$

All three dependences of $2r_M$ on $\langle\bar{u}\rangle$ are shown in Fig. 4.

As is seen from (30), (31), a "large" tube becomes "small" as the velocity changes. This is seen in Fig. 3. The drag curve when the time is the main parameter (see (20), (21)) has a slope $(A_1 + 2A_3)$ greater than the slope $(A_1 + A_3)$ delimiting the curve (26), and hence intersects it.

The minimum diameter concept was first discussed in [26], and then in [39, 45]. The formula

$$2r_M^+ \approx 60.6\Delta B \tag{32}$$

was found empirically in [45]. A rather different dependence

$$2r_M^+ \approx 250\exp(\Delta B/A_3) \tag{33}$$

follows from (11), (24).

Small Diameter Tubes. Although the molecular properties of the solution in tubes with radii r less than r_M are essential for all stream domains, nevertheless, the influence of viscosity for $r \gg \Delta_1$ will be bounded by the thin near-wall domain as before. The drag reduction is hence large ($l^+ \gg 1$) and just as has been done for "large" tubes, the following asymptotic formulas can be written

$$\begin{gathered} U^+ - \langle u^+ \rangle = \varphi_1(\eta,\ l/r,\ p),\quad l^+,\ z^+ \gg 1, \\ \langle u^+ \rangle = \varphi(z^+,\ p),\quad z^+ \ll r^+,\ l^+, \\ \varphi_1 = -A_2(p)\ln\eta + \psi(l/r,\ p),\quad 1 \ll \eta r^+ \ll l^+,\ r^+, \\ \varphi = -A_2(p)\ln z^+ + B_2(p),\quad 1 \ll z^+ \ll l^+,\ r^+. \end{gathered} \tag{34}$$

Hence, it is seen in particular that for a large drag reduction in small tubes the logarithmic distribution with coefficient A_1 is displaced by a logarithmic distribution with coefficient $A_2(p)$.

Furthermore, for $\Delta_1 \ll r$ we have the relationships

$$\begin{gathered} U^+ = A_2(p)\ln r^+ + B_2(p) + \psi(l/r,\ p), \\ U^+ - \langle \bar{u}^+ \rangle \approx C(l/r,\ p), \end{gathered}$$

which can be rewritten as a drag law

$$\sqrt{8/\lambda} = A_2(p)\ln(\mathrm{Re}\sqrt{\lambda/32}) + B_2'(l/r,\ p), \tag{35}$$

in which an unknown function of the tube radius entered.

An attempt was made in [45] at an experimental determination of a small tube drag law. However the spread of the data around the curve proposed there was sufficiently great.

Limit Asymptote. If the constraint (18) for one of the coefficients in (35) is supplemented by a constraint for B_2', then we obtain the equation of some limit curve enclosing the drag curves for solutions from below. We obtain such a constraint by assuming a lower critical transition point to turbulence in a solution exactly as in a viscous fluid. Then the equation of the limit curve passing through it will be

$$\sqrt{8/\lambda} \approx 11.6\ln(\mathrm{Re}\sqrt{\lambda/32}) - 32. \tag{36}$$

Such an equation for the limit asymptote of the drag reduction was proposed in [16, 21, 42, 43, 54]. Power-law approximations of the limit asymptote were discussed in [11, 16, 19, 20-22].

A distinctive singularity of small tubes is the fact that drag reduction can already occur therein in the transition region for $\mathrm{Re} \approx 2.3 \cdot 10^3 - 3.5 \cdot 10^3$. The upper critical Reynolds number of the transition to turbulence can hence be absent for a solution. The question of the lower critical number is not even simple. The lag of the transition in solutions is mentioned in [21, 23, 29, 48]. The lower critical Reynolds number does not vary according to [14, 16]. However, significant perturbation level in the flow of solutions has already been noted at $\mathrm{Re} \leqslant 10^3$ in many papers [6, 33, 42, 49, 65, 66].

NOTATION

$\langle u \rangle$ is the mean flow velocity;
$\langle \bar{u} \rangle$ is the fluid discharge through a tube section;

U	is the maximum velocity;
u_*	is the dynamic velocity;
$u^+ = u/u_*$	is the dimensionless velocity;
r	is the tube radius;
z	is the distance from its wall;
$\eta \equiv z/r$, $z^+ = zu_*/\nu$	are the dimensionless distances;
ν	is the viscosity coefficient;
θ	is the characteristic time parameter of a fluid;
l	is the characteristic length parameter;
$\bar{l}$, $\bar{\theta}$	are the renumbered length and time;
A_n, B_n	are the coefficients of the distributions for the velocity;
$\Delta B = B - B_0$	is the non-Newtonian increment of B;
Δ_1	is the viscous sublayer thickness;
Δ_2	is the transition zone thickness,
Re	is the Reynolds number;
λ	is the drag coefficient;
u_{*cr}	is the critical value of the dynamic velocity;
$2r_M$	is the minimum diameter;
p	is the set of dimensionless characteristics of the solution.

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